

General Parity Result and Cycle-plus-Triangles Graphs

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Abstract

We generalize a parity result of Fleishner and Stiebitz that being combined with Alon–Tarsi polynomial method allowed them to prove that a 4-regular graph formed by a Hamiltonian cycle and several disjoint triangles is always 3-choosable. Also we present a modification of polynomial method and show how it gives slightly more combinatorial information about colourings than direct application of Alon’s Combinatorial Nullstellensatz.

We start with the following parity theorem.

Theorem 1. *Let $V = \sqcup_{i=1}^n V_i$ be a finite set partitioned onto disjoint subsets V_i of odd sizes $|V_i|$. Let G be a graph on a ground set V such that each V_i is independent set in G and each bipartite subgraph induced on $V_i \sqcup V_j$ is Eulerian (i.e. all degrees are even). Consider the subsets $U \subset V$ such that $|U \cap V_i| = 1$ for all i and subgraph induced on U is Eulerian. Then the number of such U is odd.*

Proof. Consider ordered sequences of vertices $(x_1, \dots, x_n, y_1, \dots, y_n)$ so that for all $i = 1, \dots, n$:

- (i) $x_i \in V_i$;
- (ii) $y_i \in \{x_1, \dots, x_n\}$;
- (iii) either $y_i = x_i$ or x_i and y_i are joined by edge in G . Call it a special sequence.

We have to prove that number of such sequences is odd. Indeed, given x_1, \dots, x_n fixed the number of ways to choose y_1, \dots, y_n is odd if and only if the subgraph on $\{x_1, \dots, x_n\}$ is Eulerian.

For any special sequence $\lambda = (x_1, \dots, y_n)$ we construct a directed graph $G(\lambda)$ on $\{1, 2, \dots, n\}$: draw directed edge from i to $j \neq i$ if $y_i \in V_j$. This is a directed graph with outdegrees at most 1. Clearly x_1, \dots, x_n and graph G define y_1, \dots, y_n (at most) uniquely. Further in the proof a cycle of length at least 3 is called long. Denote by \mathcal{L} the set of special sequences λ for which $G(\lambda)$ has a long cycle. We prove that $|\mathcal{L}|$ is even by constructing an involution without fixed

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points on \mathcal{L} . It acts as follows. Choose a minimal (lexicographically) long cycle in G and reverse its edges. For example, if the minimal cycle is formed by edges $2 - 3, 3 - 9, 9 - 2$, i.e. $y_2 = x_3, y_3 = x_9, y_9 = x_2$, we replace (y_2, y_3, y_9) from (x_3, x_9, x_2) to (x_9, x_2, x_3) . No new cycles appear, since any edge in G may belong to at most one cycle. Hence this map is an involution without fixed points, as desired.

Thus the parity of the number of special sequences is the same as the parity of the number of special sequences λ for which graph $G(\lambda)$ does not have long cycles. Number of special sequences with empty $G(\lambda)$ equals $\prod |V_i|$, i.e. is odd. So, it suffices to prove that if a non-empty directed graph G on $\{1, \dots, n\}$ without long cycles is fixed, the number of special sequences λ with $G(\lambda) = G$ is even. It is almost obvious. Indeed, G has a vertex which has at most 1 neighbour, say, vertex p is joined only with q (by 1 or 2 edges). Fix all x_i for $i \neq p$. The number of ways to choose x_p is the number of neighbors of x_q in V_p , it is even number. \square

Corollary. *Given a circle γ . Let P_1, \dots, P_n be closed polygonal lines inscribed in γ , each having odd number of edges, without common vertices. Then the total number of ways to choose edges s_I of P_i , $i = 1, \dots, n$, so that each chosen edge intersects even number of other chosen edges, is odd.*

The following corollary is the crucial parity theorem of [1], originally proved by successive modifications of the graph.

Corollary. *Consider a 4-regular (multi)graph G on the ground set $V = \{x_1, \dots, x_{3n}\}$, $x_{3n+1} = x_1$ (we identify vertices and abstract variables), which is a union of Hamiltonian cycle $x_1 - x_2 - \dots - x_{3n} - x_1$, naturally considered as a regular $3n$ -gon, and n triangles $a_i - b_i - c_i - a_i$, $i = 1, \dots, n$. Consider the following Laurent polynomial*

$$\Phi(x_1, \dots, x_{3n}) = \prod_{i=1}^{3n} (1 - x_{i+1}/x_i) \prod_{i=1}^n (1 - a_i/b_i)(1 - b_i/c_i)(1 - c_i/a_i).$$

Then the constant term $CT[\Phi]$ is congruent to 2 modulo 4.

Proof. Start with expanding brackets in $\prod_{i=1}^{3n} (1 - x_{i+1}/x_i)$. We get monomials with each variable in a power 0 or ± 1 , and powers +1 and -1 alternate. Coefficient of each such monomial is ± 1 .

Now consider triangles. We have $(1 - a/b)(1 - b/c)(1 - c/a) = a/c + c/b + b/a - c/a - b/c - a/b$. That is, for any triangle we should take one vertex in power 1, another in power -1, third in power 0. Draw an arrow $a \rightarrow b$ if we choose $-a/b$, and so on. Additionally colour a in black and b in white. So, we have one black-to-white arrow for each triangle. Product of corresponding multiples may cancel (i.e. give a constant product) with the unique multiple arising from $\prod_{i=1}^{3n} (1 - x_{i+1}/x_i)$. This happens if only if black and white vertices alternate. This in turn happens exactly when each chosen arrow intersects even number of other chosen arrows. And for given n not-oriented edges there exist exactly two ways to draw arrows on them in such a way that black and white vertices alternate. These two ways give a total amount +2 or -2 to the constant term of Φ (since they are obtained one from another by changing summand in all $6n$ brackets). Now we just use the previous corollary and conclude that half of the constant term of Φ is odd. \square

Now we write down the formula for the central coefficient of Φ via the values of Φ on a grid. Choose sets A_1, \dots, A_{3n} of cardinality 3 in $K \setminus \{0\}$ for some field K (we use only $K = \mathbb{R}$, but it is possible that other fields may be useful for other goals.) Define a function φ_i on the set A_i . If, say, $A_i = \{u, v, w\}$ we put $\varphi_i(u) = \frac{vw}{(u-v)(u-w)}$ and so on. Then

$$\sum_{x \in A_i} \varphi_i(x) x^d = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d = 1 \text{ or } d = 2. \end{cases} \quad (1)$$

The formula for the constant term is

$$CT[\Phi] = \sum_{x_i \in A_i} \Phi(x_1, \dots, x_{3n}) \cdot \prod_{i=1}^{3n} \varphi_i(x_i). \quad (2)$$

Indeed, expand $\Phi(x_1, \dots, x_{3n})$. For each monomial term $\prod x_i^{d_i}$ in Φ the sum of products is a product of sums:

$$\sum_{x_i \in A_i} \prod_{i=1}^{3n} x_i^{d_i} \cdot \prod_{i=1}^{3n} \varphi_i(x_i) = \prod_{i=1}^{3n} \left(\sum_{x \in A_i} \varphi_i(x) x^{d_i} \right)$$

Equation (1) yields that this product equals 1 if $d_1 = \dots = d_{3n} = 0$ (for the constant term) and equals 0 if $d_i \in \{1, 2\}$ for at least one index i (it happens for each non-constant term of Φ , that is seen from the formula for Φ). This gives (2).

Immediate corollary of (2) is that Φ can not vanish on $\prod A_i$. In other words, graph G is 3-choosable.

Now consider genuine colourings of G . Assume that G is properly 3-coloured, we identify colours with three real numbers u, v, w . Clearly vertices of any triangle have different colours, therefore there are n vertices of each colour. Denote by U, V, W number of vw -edges, uw -edges, uv -edges respectively in the Hamiltonian cycle. Then $U + V$ is twice more than the number of w -vertices, i.e. $U + V = 2n$, analogously $V + W = U + W = 2n$, hence $U = V = W = n$.

Now we take $A_1 = A_2 = \dots A_{3n} = A = \{u, v, w\}$ and apply formula (2). Consider non-zero summand in RHS of (2), it corresponds to some proper colouring. We have

$$\begin{aligned} \prod_{i=1}^{3n} \varphi_i(x_i) &= (-1)^{3n} \frac{(uvw)^{2n}}{(u-v)^{2n}(v-w)^{2n}(w-u)^{2n}} \\ \prod_{i=1}^n (1 - a_i/b_i)(1 - b_i/c_i)(1 - c_i/a_i) &= \pm \frac{(u-v)^n(v-w)^n(w-u)^n}{(uvw)^n} \\ \prod_{i=1}^{3n} (1 - x_{i+1}/x_i) &= \pm \frac{(u-v)^n(v-w)^n(w-u)^n}{(uvw)^n}. \end{aligned}$$

Totally

$$\Phi(x_1, \dots, x_{3n}) \prod_{i=1}^{3n} \varphi_i(x_i) = \pm 1.$$

Let's see what happens if we rename colours. If we simultaneously replace, say, colours u and v , we totally change sign of $2n$ or $6n$ multiples, hence sign of the product does not change. Therefore we may partition all non-zero summands in the RHS of (2) onto 6-tuples with the same value of summands, and the number of 6-tuples equals the number of essentially different 3-colourings of G (permutation of colours gives the same colouring). Hence we have proved

Theorem 2. *Number of essentially different 3-colourings of the cycle-plus-triangles graph G is odd.*

Corollary. *There exists a proper colouring of G in 3 colours white, blue and red such that blue and red vertices form a connected graph.*

Proof. Assume the contrary. Then for given n white vertices, other $2n$ vertices have $r \geq 2$ connected components. We may interchange blue and red colour in each component by all possible 2^r ways. Therefore total number of white-blue-red colourings is divisible by 4, on the other hand, it is 6 times more than the odd number of essentially different colourings. The contradiction. \square

Above modification of polynomial method is essentially the same as proposed in [3, 4], the only difference is that we apply to it Laurent polynomials directly, without making polynomials from them. In [1] the authors used the method of [2], which was later explained as application of Combinatorial Nullstellensatz in the main survey by Alon [5].

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